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Abstract

We consider quantum walks on the cycle in the non-stationary case where the 'coin' operation is allowed to change at each time step. We characterize, in algebraic terms, the set of possible state transfers and prove that, as opposed to the stationary case, the associate probability distribution may converge to a uniform distribution among the nodes of the associated graph.

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1. Non-stationary quantum walks on the cycle

Consider a non-oriented graph where all the N nodes have the same degree d and assume that, at each time step, a walker makes a choice, out of a set of d elements, $\{1, \ldots, d\}$, a *(generalized) coin*, with probability $p_1, p_2, \ldots, p_d$, respectively. The walker starts from a given node of the graph and moves in a direction determined by the choice in $\{1, \ldots, d\}$. After time t, the walker will have a probability $P(j, t)$ of being found in the node j, $j = 1, \ldots, N$. Such a system is known as a *random walk* on a graph. A *quantum walk* is the quantum counterpart of a random walk in that both the walker and the coin are seen as quantum systems of dimensions N and d, respectively. At each step an operation C is performed on the coin system and then an operation is performed on the walker system. The latter operation depends on the state of the coin system.

Quantum walks have recently received large attention due to the fact that they can model quantum algorithms and generate interesting quantum states. There are several review papers on quantum walks, their use, dynamics, implementations and generalizations (see, e.g., [10, 11]). In most studies presented so far, the coin operation C is fixed and repeated at each time step. We shall call this type of quantum walks *stationary*, while quantum walks where the coin operation is allowed to change at each time step will be named *non-stationary*. Studies exist on how the parameters of C affect the behavior of the quantum walk [19]. The

non-stationary case has been considered in both numerical and analytic studies where the coin operation is allowed to change at each step according to a prescribed sequence or it is random [12, 13, 15, 17]. It is shown that, for certain walks, the presence of random noise in C at each step, the so-called *unitary noise*, causes a behavior similar to the classical random walk. A non-stationary setting can be considered also for other types of random walks such as classical walks on groups (as for example the Heisenberg group) [7, 8, 14]. These systems are of current interest as models of quantum dynamics. The role of the coin process is played by a dynamical system which may be characterized by a time-varying transformation therefore giving rise to a non-stationary random walk on a group. Similar questions to the ones treated here, in particular concerning the set of achievable distributions, can be asked in that setting as well.

In this paper, we approach the study of non-stationary quantums walk from the point of view of *design* and *control* [5]. We consider the coin operation C as a control variable which we can change at each step in order to obtain a desired behavior. The first questions that arise in this setting are therefore about the type of behavior that can be obtained (in particular the type of probability distributions) and whether there are significant differences with the stationary case. This paper is a first study in this direction.

Current proposals for implementations of stationary quantum walks (see [10] and references therein) may be modified in order to obtain a non-stationary walk. This is discussed for example in [15] for a specific experimental proposal where a variable coin operation can be obtained by varying the duration of a laser pulse.

The quantum walk on the cycle is the simplest finite-dimensional quantum walk. The study of stationary quantum walks on the cycle was started in [1]. In this case, as a consequence of the reversibility of the evolution, the probability distribution $P(j, t)$ does not converge to a constant value as $t \to \infty$. This is in contrast with the classical random walk on the cycle whose probability distribution converges to a uniform distribution independently of the initial state. For this reason, a Cesaro type of alternative probability distribution is introduced which is the average of $P(j, t)$ over an interval of time $[0, t]$. With this definition, a uniform limit distribution is obtained for a number of positions N odd, which is independent of the initial state as long as this one is localized in one given position. For N even, there is a much richer behavior and different limit distributions are obtained for different initial states as discussed in [3, 4]. From an experimental point of view having a uniform Cesaro-type limit probability distribution means that there is equal probability of finding the walker in one of the positions by measuring at a random time over a very large interval.

In dealing with non-stationary walks on the cycle, the first question concerns the type of (non-Cesaro) probability distributions that can be obtained. This question is of interest in the use of random walks for algorithmic purposes. In fact, there exist several computational algorithms which are based on sampling from a given set of objects according to a prescribed distribution [18]. These algorithms are referred to as *randomized algorithms*. If one uses a quantum walk to implement one of these algorithms, one can obtain the desired sample by measuring the position of the walker. One natural question concerns the set of possible distributions available. We shall answer this question in Lie algebraic terms in this paper for a non-stationary walk on the cycle and will show that it is possible that the probability distribution converges to a *uniform* distribution. There are at least two reasons to consider the uniform distribution with special attention. One is that it offers an example of a *limit* distribution (in the non-Cesaro sense) which is not available in the stationary case. In fact, we will show that it is possible to reach a separable state of the form $|1\rangle \otimes |w\rangle$, where $|1\rangle$ is a state of a two-dimensional coin and $|w\rangle$ is a state of the walker with all equiprobable positions (cf formulas (17) and (18) below). Since the coin operation is arbitrary, we can

set it equal to the identity in the following steps, and in the following evolution the walker will just move from one position to the other in a cycle and the probability of each position will be uniform. The probability distribution therefore reaches the uniform value and then stays constant. This is an example of a behavior different from the stationary case. The second reason to consider the uniform distribution in more detail is that this is the limit of the corresponding classical random walk on the cycle. Therefore, if an algorithm uses this feature of the classical counterpart, it can be implemented with the non-stationary quantum walk. For example, a randomized algorithm which requires sampling from a uniform distribution can be implemented by measuring the position of the walker at a large time. If we use the Cesaro definition of probability distribution we could perform a measurement but will have to select (again) a random time over a large interval.

A quantum walk on the cycle is a bipartite quantum system $\mathcal{C} \otimes \mathcal{W}$, where the system $\mathcal{C}$, the coin, is a two-level system with orthonormal basis states $|{+}1\rangle$ and $|{-}1\rangle$. The system $\mathcal{W}$, the walker, is an N-level system with orthonormal basis states $|0\rangle, |1\rangle, \ldots, |N-1\rangle$. At the tth time step, one performs a coin operation of the form $C_t \otimes \mathbf{1}$, where C_t is an arbitrary (special) unitary operation on the two-dimensional Hilbert space associated to $\mathcal{C}$, i.e., an element of $SU(2)$. This is followed by a conditional shift S on the Hilbert space associated with $\mathcal{W}$ defined as

$$S|c\rangle \otimes |j\rangle = |c\rangle \otimes |(j+c) \bmod N\rangle.$$

By considering the standard basis $|e_j\rangle$, $j = 1, \ldots, 2N$, defined by $|e_j\rangle := |1\rangle \otimes |j-1\rangle$ and $|e_{j+N}\rangle := |-1\rangle \otimes |j-1\rangle$, $j = 1, \ldots, N$, the matrix representation of the operator $C_t \otimes \mathbf{1}$ is $C_t \otimes \mathbf{1}_{N \times N}$, where $\mathbf{1}_{N \times N}$ is the $N \times N$ identity,[4] $\otimes$ denotes the Kronecker product of matrices and $C_t \in SU(2)$. The matrix representation of the operator S is the block diagonal matrix $\mathrm{diag}(F, F^T)$, where F is the basic circulant permutation matrix, that is,

$$S := \mathrm{diag}(F, F^T), \qquad F := \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\ 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 & 0 \end{pmatrix}. \tag{1}$$

The probability of finding the walker in the state $|j-1\rangle$, $j = 1, \ldots, N$, is the sum of the probabilities of finding the state of the composite system $\mathcal{C} \otimes \mathcal{W}$ in $|1\rangle \otimes |j-1\rangle := |1, j-1\rangle$ and $|-1\rangle \otimes |j-1\rangle := |-1, j-1\rangle$. That is, if $|\psi\rangle$ is the state of the composite system,

$$P(j-1, t) = |\langle \psi(t)|1, j-1\rangle|^2 + |\langle \psi(t)|-1, j-1\rangle|^2$$
$$= |\langle \psi(t)|e_j\rangle|^2 + |\langle \psi(t)|e_{j+N}\rangle|^2. \tag{2}$$

By writing

$$|\psi\rangle := \sum_{k=1}^{2N} \alpha_k |e_k\rangle, \qquad \sum_{k=1}^{2N} |\alpha_k|^2 = 1, \tag{3}$$

we have

$$P(j-1, t) = |\alpha_j|^2 + |\alpha_{j+N}|^2, \qquad j = 1, \ldots, N. \tag{4}$$

In what follows we shall make the following standing assumption.

Assumption. N is an *odd* number.

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[4] We replace this notation by $\mathbf{1}$ when there is no ambiguity on the dimension.

We use this technical assumption in several steps and in particular in theorem 1 to show that the matrix S defined in (1) is in a certain Lie group (cf formula (11)). In the stationary case, problems with the N even case arise from the degeneracy of eigenvalues in the basic $S(C \otimes \mathbf{1})$ operation. As we have mentioned, this leads to a very different and richer behavior with respect the N odd case.

2. Characterization of the admissible evolutions

In this section, we characterize the set of unitary evolutions available for a non-stationary quantum walk on the cycle, that is, the set of available state transfers. This is the set of finite products of operators of the form $S(C \otimes \mathbf{1}_{N \times N})$, where S is defined in (1) and $C \in SU(2)$. We denote such a set by **G**. The set **G** is a group. It is in fact a Lie group as is shown in the following theorem. In order to state this theorem, we need to recall some properties of circulant matrices [6] and define two Lie algebras.

Circulant $N \times N$ matrices with complex entries form a vector space over the real numbers. Each matrix is determined by the first row since all the other rows can be obtained by cyclic permutation of the first one. Moreover, every complex circulant matrix R can be written as linear combination with complex coefficients of the basic permutation matrix F defined in (1) and its powers from 0 to $N - 1$, i.e.,

$$R := \sum_{l=0}^{N-1} a_l F^l, \tag{5}$$

with N complex coefficients $a_0, \ldots, a_{N-1}$. All the circulant matrices commute. If we require that R is not only circulant but also skew-Hermitian then we must have

$$R^\dagger = a_0^* \mathbf{1} + \sum_{l=1}^{N-1} a_l^* F^{lT} = -R = -a_0 \mathbf{1} - \sum_{l=1}^{N-1} a_l F^l, \tag{6}$$

and with a change of index $l \to N - l$ and using $F^{N-l^T} = F^l$, we have

$$R^\dagger = a_0^* \mathbf{1} + \sum_{l=1}^{N-1} a_{N-l}^* F^l = -a_0 \mathbf{1} - \sum_{l=1}^{N-1} a_l F^l. \tag{7}$$

This gives the relations

$$a_0^* = -a_0, \qquad a_{N-l}^* = -a_l, \qquad l = 1, \ldots, \frac{N-1}{2}. \tag{8}$$

Equations (7) constitute N-independent relations on the $2N$ real parameters of R and show that the space of skew-Hermitian circulant matrices is a real vector space of dimension N.

Denote by $\mathcal{L}$ the Lie algebra spanned by the $2N \times 2N$ skew-Hermitian matrices of the form

$$L_1 := \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \qquad \text{and} \qquad L_2 := \begin{pmatrix} 0 & Q \\ -Q^\dagger & 0 \end{pmatrix}, \tag{9}$$

with R being a skew-Hermitian circulant $N \times N$ matrix and Q a general circulant matrix. It is easily seen that this is in fact a Lie algebra of (real) dimension $3N$; the fact that it is closed under the Lie bracket being a consequence of the fact that the product of two circulant matrices is another circulant matrix. Note, in particular, that matrices of the type L_1 form an Abelian subalgebra of dimension N. We denote by $e^{\mathcal{L}}$ the connected Lie group associated with $\mathcal{L}$.

Theorem 1. *The set* **G** *of possible evolutions of a non-stationary quantum walk is the Lie group* $e^{\mathcal{L}}$.

Proof. We define an auxiliary Lie algebra $\mathcal{L}'$, prove that $\mathbf{G} = e^{\mathcal{L}'}$ and then prove that $\mathcal{L} = \mathcal{L}'$. The claim then follows from the correspondence between Lie groups and Lie algebras. We denote by $\mathcal{L}'$ the Lie algebra generated by the set

$$\mathcal{F} := \{su(2) \otimes \mathbf{1}, S(su(2) \otimes \mathbf{1})S^T, \dots, S^{N-1}(su(2) \otimes \mathbf{1})S^{(N-1)T}\}, \tag{10}$$

where S is defined in (1) and T denotes the transposition.

To show that $\mathbf{G} \subseteq e^{\mathcal{L}'}$, it is enough to show that both $C \otimes \mathbf{1}$, $C \in SU(2)$ and S are in $e^{\mathcal{L}'}$. This fact is obvious for $C \otimes \mathbf{1}$, since this is the exponential of an element in $su(2) \otimes \mathbf{1}$. For S, we consider the elements $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}$ and $S^{\frac{N-1}{2}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1}\right)S^{(\frac{N-1}{2})T}$, both in $e^{\mathcal{L}'}$, and calculate with (1)

$$\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{1}\right]\left[S^{\frac{N-1}{2}}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \mathbf{1}\right)S^{(\frac{N-1}{2})T}\right]$$
$$= \begin{pmatrix} F^{(N-1)T} & 0 \\ 0 & F^{N-1} \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & F^T \end{pmatrix} := S. \tag{11}$$

We have used $F^{(N-1)T} = F$.

To show that $e^{\mathcal{L}'} \subseteq \mathbf{G}$ it is enough to show that every element of the type $S^j(X \otimes \mathbf{1})S^{jT}$, with $X \in SU(2)$, $j = 0, \dots, N-1$, can be written as the finite product of elements of the form $S(C \otimes \mathbf{1})$ with $C \in SU(2)$.[5] This is readily seen because, with $X \in SU(2)$, for every j,

$$S^j(X \otimes \mathbf{1}_{N \times N})S^{jT} = (S^j(X \otimes \mathbf{1}_{N \times N}))(S^{N-j}(\mathbf{1}_{2 \times 2} \otimes \mathbf{1}_{N \times N})). \tag{12}$$

To conclude the proof, we show that $\mathcal{L} = \mathcal{L}'$ showing that $\mathcal{F} \subseteq \mathcal{L}$ and a basis of $\mathcal{L}$ can be obtained as (repeated) Lie brackets and/or linear combinations of elements of $\mathcal{F}$ in (10). A general matrix in $\mathcal{F}$ has the form, with $A \in su(2)$,

$$S^j(A \otimes \mathbf{1}_{N \times N})S^{jT} = \begin{pmatrix} F^j & 0 \\ 0 & F^{jT} \end{pmatrix}\begin{pmatrix} ib\mathbf{1}_{N \times N} & \alpha\mathbf{1}_{N \times N} \\ -\alpha^*\mathbf{1}_{N \times N} & -ib\mathbf{1}_{N \times N} \end{pmatrix}\begin{pmatrix} F^{jT} & 0 \\ 0 & F^j \end{pmatrix}$$
$$= \begin{pmatrix} ib\mathbf{1}_{N \times N} & \alpha F^{2j} \\ -\alpha^*(F^{2jT}) & -ib\mathbf{1}_{N \times N} \end{pmatrix}, \tag{13}$$

with general b real and α complex, $j = 0, \dots, N-1$. This is clearly in $\mathcal{L}$. Elements of the form L_2 in (9) are real linear combinations of elements of the form $\begin{pmatrix} 0 & \gamma F^k \\ -\gamma^* F^{kT} & 0 \end{pmatrix}$ which are of the form in (13) with $b = 0$, $\gamma = \alpha$ and $j = \frac{k}{2}$ for k even and $j = \frac{N+k}{2}$ for k odd. A basis for the real elements of the type L_1 is given by the $\frac{N-1}{2}$ linearly independent elements

$$\begin{pmatrix} F^j - F^{jT} & 0 \\ 0 & -(F^j - F^{jT}) \end{pmatrix}, \qquad j = 1, \dots, \frac{N-1}{2}. \tag{14}$$

These are obtained as Lie brackets of $\begin{pmatrix} 0 & \mathbf{1}_{N \times N} \\ -\mathbf{1}_{N \times N} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & F^j \\ -F^{jT} & 0 \end{pmatrix}$ which are both of type L_2. A basis for the purely imaginary elements of type L_1 is given by the $\frac{N+1}{2}$ linearly independent elements of the type

$$\begin{pmatrix} i(F^j + F^{jT}) & 0 \\ 0 & -i(F^j + F^{jT}) \end{pmatrix}, \qquad j = 0, \dots, \frac{N-1}{2}, \tag{15}$$

which are obtained as Lie brackets of $\begin{pmatrix} 0 & F^j \\ -F^{jT} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i\mathbf{1}_{N \times N} \\ i\mathbf{1}_{N \times N} & 0 \end{pmatrix}$. This completes the proof of the theorem. $\qquad\square$

---

[5] Recall that every element of a connected Lie group can be obtained as the finite product of exponentials of a set of generators of the corresponding Lie algebra (see, e.g., [9]) and the exponential map is surjective on $SU(2)$ (see, e.g., [16]).

3. Obtaining the uniform distribution

The Lie group $\mathbf{G} = e^{\mathcal{L}}$, having dimension $3N$, is not isomorphic, for $N \geqslant 3$, to $SU(2N)$ (which has dimension $4N^2 - 1$) nor to $Sp(N)$ (which has dimension $N(2N + 1)$). Therefore, $\mathbf{G} = e^{\mathcal{L}}$ is not transitive on the complex sphere of dimension $2N$ which means that there are state transfers for the quantum system of coin and walker which are not induced by any transformation in $\mathbf{G}$ [2]. Some state transfers are of special interest. In particular, we are interested in whether a state of the form

$$|\psi_{\text{in}}\rangle := |\psi_{\text{coin}}\rangle \otimes |0\rangle, \tag{16}$$

that is, a state corresponding to the walker with certainty in position $|0\rangle$, can be transferred to a state corresponding to the uniform distribution. This is a state where the probability $P(j - 1, t)$ in (2) is equal to $\frac{1}{N}$, for every $j = 1, \ldots, N$, at some t, that is, the walker is found in any position with the same probability. Since, $\forall C \in SU(2)$, $C \otimes \mathbf{1}_{N \times N} \in e^{\mathcal{L}}$, we can assume, without loss of generality, that $|\psi_{\text{coin}}\rangle$ in (16) is $|1\rangle$ so that the problem is to transfer the state $|e_1\rangle := [1, 0, \ldots, 0]^T$ to a state with the desired property. We shall show in the following that such a state transfer is possible.

Theorem 2. *There exists a matrix L in $\mathcal{L}$ such that*

$$e^L |e_1\rangle = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{17}$$

where

$$|r_1|^2 = |r_2|^2 = \cdots = |r_N|^2 = \frac{1}{N}. \tag{18}$$

In order to prove this theorem we first prove a lemma. Recall the definition of the Fourier matrix Φ of order N (see, e.g., [6]). This is defined so that its conjugate transposed is

$$\Phi^{\dagger} := \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \ldots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \ldots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \ldots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \ldots & \omega^{(N-1)(N-1)} \end{pmatrix}, \tag{19}$$

where ω is the Nth root of the unity, that is $\omega := e^{i\frac{2\pi}{N}}$. The Fourier matrix Φ is unitary.

Lemma 3.1. *Define*

$$x_l := \frac{l(l-1)\pi}{N}, \qquad l = 0, 1, \ldots, N - 1. \tag{20}$$

Then

$$
\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_N \end{pmatrix} := \frac{1}{\sqrt{N}} \Phi^\dagger \begin{pmatrix} e^{ix_0} \\ e^{ix_1} \\ e^{ix_2} \\ \vdots \\ e^{ix_{N-1}} \end{pmatrix} \tag{21}
$$

has the property (18).

Proof. From (19) and (21), we obtain

$$
r_h = \frac{1}{N} \left(1 + \sum_{l=1}^{N-1} \omega^{(h-1)l} e^{ix_l} \right), \qquad h = 1, \ldots, N. \tag{22}
$$

This, using the definition of ω, gives

$$
r_h = \frac{1}{N} \left(1 + \sum_{l=1}^{N-1} e^{\frac{i2\pi(h-1)l}{N}} e^{ix_l} \right). \tag{23}
$$

We calculate $|r_h|^2$, $h = 1, \ldots, N$, as

$$
|r_h|^2 = r_h^* r_h = \frac{1}{N^2} \sum_{l_1, l_2=0}^{N-1} e^{i\frac{2\pi}{N}(l_2-l_1)(h-1)} e^{i(x_{l_2}-x_{l_1})}
$$

$$
= \frac{1}{N} + \frac{2}{N^2} \sum_{\{l_1, l_2\}=0}^{N-1} \mathrm{Re}\!\left(e^{i\frac{2\pi}{N}(l_2-l_1)(h-1)} e^{i(x_{l_2}-x_{l_1})} \right). \tag{24}
$$

The sum in the last term is intended over all the pairs of indices $\{l_1, l_2\}$, with $l_1 \neq l_2$, where only one is chosen between $\{l_1, l_2\}$ and $\{l_2, l_1\}$. Because of the presence of the real part 'Re' it is not important which pair is chosen. We now show that, with the choice (20), the last term of this expression is zero for every h, which will prove the claim that $|r_h|^2 = \frac{1}{N}$.

It is convenient to re-write the sum by regrouping elements corresponding to $l_2 - l_1 = p \bmod N$, for $p = 1, \ldots, N-1$. This means $l_2 - l_1 = p$ or $l_1 - l_2 = N - p$. We have

$$
\sum_{\{l_1, l_2\}=0}^{N-1} \mathrm{Re}\!\left(e^{i\frac{2\pi}{N}(l_2-l_1)(h-1)} e^{i(x_{l_2}-x_{l_1})} \right)
$$

$$
= \sum_{p=1}^{N-1} \mathrm{Re} \left(\sum_{l_2-l_1=p} e^{i(l_2-l_1)(h-1)\frac{2\pi}{N}} e^{i(x_{l_2}-x_{l_1})} + \sum_{l_1-l_2=N-p} e^{i(l_2-l_1)(h-1)\frac{2\pi}{N}} e^{i(x_{l_2}-x_{l_1})} \right). \tag{25}
$$

Doing the substitution $l_1 = l$ and $l_2 = l + p$ in the first term of the sum and the substitution $l_1 = l$ and $l_2 = l - (N - p)$ in the second term, this sum becomes

$$
\sum_{p=1}^{N-1} \mathrm{Re} \left(e^{ip(h-1)\frac{2\pi}{N}} \left(\sum_{l=0}^{N-1-p} e^{i(x_{l+p}-x_l)} + \sum_{l=N-p}^{N-1} e^{i(x_{l-(N-p)}-x_l)} \right) \right). \tag{26}
$$

We now show that, with the choice (20), the content of the innermost parenthesis in the above expression, i.e.,

$$
M := M(p) := \sum_{l=0}^{N-1-p} e^{i(x_{l+p}-x_l)} + \sum_{l=N-p}^{N-1} e^{i(x_{l-(N-p)}-x_l)}, \tag{27}
$$

is zero for each p which will conclude the proof of the lemma. Replacing (20) in (27) and after some algebraic manipulations, we obtain

$$M(p) = \sum_{l=0}^{N-1-p} e^{i\frac{2\pi}{N}(\frac{p(p-1)}{2}+pl)} + \sum_{l=N-p}^{N-1} e^{i\frac{2\pi}{N}(\frac{(N-p)(N-p+1)}{2}-(N-p)l)} = \sum_{l=0}^{N-1} e^{i\frac{2\pi}{N}(\frac{p(p-1)}{2}+pl)}. \tag{28}$$

Thus, we have

$$M(p) = e^{i\frac{2\pi}{N}\frac{p(p-1)}{2}} \sum_{l=0}^{N-1} e^{i\frac{2\pi pl}{N}} = e^{i\frac{2\pi}{N}\frac{p(p-1)}{2}} \frac{1-e^{i2\pi p}}{1-e^{i\frac{2\pi p}{N}}} = 0 \quad \forall\, p \neq 0 \bmod N. \tag{29}$$

This concludes the proof of the lemma. $\qquad\square$

We are now ready to prove theorem 2

Proof of theorem 2. We choose L as a matrix of the form L_1 in (9) so that e^L has the form

$$e^L = \begin{pmatrix} e^R & 0 \\ 0 & e^{-R} \end{pmatrix}, \tag{30}$$

with R being a general skew-Hermitian $N \times N$ circulant matrix. The problem is therefore to find a circulant matrix R so that

$$e^R \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix}, \tag{31}$$

with $r_1, \ldots, r_N$ satisfying (18). Any circulant matrix R is diagonalized by the Fourier matrix (19) of the corresponding dimension, that is,

$$R = \Phi^\dagger \Lambda \Phi, \tag{32}$$

with Λ being diagonal. Conversely, every matrix of the form on the right-hand side is circulant [6]. Moreover, if $\Lambda = \mathrm{diag}(i\lambda_0, i\lambda_1, \ldots, i\lambda_{N-1})$, with $\lambda_l, l = 0, \ldots, N-1$ real numbers, R is skew-Hermitian. In this case, we have

$$e^R \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \Phi^\dagger e^\Lambda \Phi \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \Phi^\dagger e^\Lambda \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \Phi^\dagger \frac{1}{\sqrt{N}} \begin{pmatrix} e^{i\lambda_0} \\ e^{i\lambda_1} \\ \vdots \\ e^{i\lambda_{N-1}} \end{pmatrix}. \tag{33}$$

Choosing $\lambda_l = x_l, l = 0, \ldots, N-1$ with definition (20), the theorem follows from lemma 3.1. $\qquad\square$

Other states with the same property can be obtained by applying a transformation $U \otimes \mathbf{1}, U \in SU(2)$, which is in **G**. In particular, note that the state (17) is a separable state.

4. Conclusion

Non-stationary quantum walks have properties which distinguish them from stationary ones. Moreover, they are amenable of study with the methods of quantum control. In fact, several

problems, such as obtaining a given evolution, can be seen as control problems where the evolution of the coin plays the role of the control. In this paper we have shown that, opposite to the stationary case, a non-stationary quantum walk on the cycle may converge to a constant distribution and in particular to a uniform distribution as for classical random walks. A constructive approach to achieve this and other evolutions of interest for general quantum walks will be the subject of future research.

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